Harmonic and Dissonant Singularities

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1. Introduction

The singular locus of a complex analytic space germ can be equipped with various analytic structures. If X is a non-smooth hypersurface $f = 0$ in some smooth germ P we denote by Sing X (resp. Sing^*X) the subspace of X of ideal $(f) + j(f)$ (resp. $(f) + m_P \cdot j(f)$) in $\mathcal{O}_P$. Here m_P is the maximal ideal of $\mathcal{O}_P$ and $j(f)$ is the jacobian ideal of f.

One knows that the analytic type of X is completely determined by Sing^*X and the dimension, see [G-H]. For Sing X this is also true for almost all X. But there do exist hypersurfaces whose analytic type cannot be revealed from the knowledge of Sing X. This "shy" species of singularities shall present our object of investigation.

The analytic and the singular stratum of a hypersurface X are the maximal subspaces A and Σ of X along which X, resp. Sing X, is trivial. Clearly $A \subset \Sigma$. X is called harmonic if $A = \Sigma$, otherwise dissonant. The harmonic singularities are exactly those which are determined by Sing X.

The main result of this paper says that dissonance is actually an infinitesimal phenomenon: X in P is called infinitesimally trivial along a smooth germ $Z \subset P$ if there exist a germ $Y \subset P$ trivial along Z and a smooth germ $H \subset P$ transversal to Z of complementary dimension such that $X \cap H^2 = Y \cap H^2$, where H^2 is the first infinitesimal neighborhood of H in P. X is called infinitesimally harmonic if it is infinitesimally trivial along Σ.

<u>Theorem.</u> A hypersurface X is harmonic if and only if it is infinitesimally harmonic.

Moreover we present a Bertini type theorem on the singular stratum of a hypersurface, and prove that most quasi-homogeneous hypersurfaces are harmonic.

2. Triviality and Constancy

Let us start with a summary of results of Rossi and Ephraim characterizing the decomposition of an analytic space germ into the product of some germ by a smooth germ. For a germ $X = (X,0)$ we denote by $A = A_X$ the germ at 0 of the set $\{x \in X, (X,x) \simeq (X,0)\}$. By Ephraim [E, Th.0.2.], A is the germ of a manifold (possibly of dimension zero). We call it the analytic stratum of X. If $Z \subset A$ we say that X is trivial along Z. Two smooth subspaces H and Z of a smooth germ P are called transversal if $T_oH + T_oZ = T_oP$ and strictly transversal if the sum is direct.

Theorem 1 (Rossi, Ephraim). For a germ X defined in a smooth P by the ideal I_X and strictly transversal smooth subspaces H and Z of P the following conditions are equivalent:

(1) X is trivial along Z.

(1') X is trivial along any smooth curve C in Z.

(2) $Z \subset X$ and the pair (X,Z) is isomorphic to $(X_o \times Z, 0 \times Z)$ for some germ X_o.

(3) $Z \subset X$ and the pair (X,Z) is isomorphic to $((X \cap H) \times Z, 0 \times Z)$.

(4) There exist vectorfields $D_{k+1},\dots,D_n$ on P tangent to Z such that $D_{k+1}(0),\dots,D_n(0)$ span T_oZ and $D_i(I_X) \subset I_X$ for all i.

(4') For any (some) coordinates (y,z) on P with $I_Z = (y_1,\dots,y_k)$ one has for all i:

$$\partial_{z_i} f \in I_X \cdot \mathcal{O}_P^m + I_Z \cdot j(f),$$

where $f \in \mathcal{O}_P^m$ is a vector whose components generate I_X and $j(f)$ is the submodule of $\mathcal{O}_P^m$ generated by the $\partial_{y_i} f$ and $\partial_{z_i} f$. If X is a hypersurface, the condition becomes: $\partial_{z_i} f \in (f) + I_Z \cdot j(f)$.

(5) There exists a projection $\pi : P \to H$ with $\pi(Z) = 0$ and $X = \pi^{-1}(X \cap H)$.

(5') There exist coordinates (y,z) on P with $I_Z = (y_1,\dots,y_k)$, $I_H = (z_{k+1},\dots,z_n)$ and such that $\partial_{z_i}(I_X) \subset I_X$ for all i.

In this last case we say that X is constant along Z w.r.t. (y,z). The equivalence (1) ⇔ (2) is [E, Th.0.2.], (2) ⇔ (3) is [E, remark following Lemma 1.5.], (2) ⇔ (4) is [R, Th.3.2.] and in the non-reduced case [F, 2.12.], (4) ⇔ (4') is a simple computation, (1) ⇔ (1') and (3) ⇔ (5) ⇒ (5') ⇒ (4) are obvious.

3. Harmony and Dissonance

In this section X will always be a non-smooth hypersurface germ. For a minimal embedding $X \subset P$ let f be a generator of $I_X \subset \mathcal{O}_P$. The singular subspace Sing X of X is the germ defined by the ideal $Tf = (f) + j(f) \subset \mathcal{O}_P$. Clearly Sing X is independent of the chosen f; and it is independent up to isomorphism of the embedding $X \subset P$. The analytic stratum $\Sigma = \Sigma_X = A_{\text{Sing } X}$ of Sing X is called the singular stratum of X. One has the inclusions $A_X \subset \Sigma_X \subset X$.

Definition. X is called harmonic if $A_X = \Sigma_X$, and harmonic along $Z \subset X$ if $A_X \cap Z = \Sigma_X \cap Z$. Otherwise X is called dissonant, resp. non-harmonic along Z. We say that X is dissonant along Z if $A_X \cap Z = 0$ and $\Sigma_X \cap Z = Z$. This is stronger than being non-harmonic along Z.

Isolated hypersurface singularities are harmonic. Examples of dissonant singularities are given below. Our interest in harmonic singularities stems from the following theorem of Gaffney and Hauser [G-H] which in the special case of an isolated singularity was earlier proven by Mather and Yau [M-Y].

Theorem 2. Two harmonic hypersurfaces X and Y of the same dimension are isomorphic if and only if their singular subspaces Sing X and Sing Y are isomorphic. The assertion fails for dissonant singularities.

The first step in studying harmonic singularities is to give an explicit criterion of harmony. For this purpose we need a version of Nakayama's Lemma and some auxiliary notation:

Lemma. Let I and J be two ideals of $\mathcal{O}_P$ generated by $f_1,\dots,f_m$ and $g_1,\dots,g_m$ respectively. If $g_i - f_i \in m_P \cdot I$ for all i, then $I = J$.

Proof. Let K denote the ideal generated by the $g_i - f_i$'s. Then $J \subset I + K \subset \subset I + m_P \cdot I \subset I \subset J + K \subset J + m_P \cdot I$ and Nakayama's Lemma implies equality.

Let now Z and H be smooth and strictly transversal in P and let (y,z) be coordinates on P with $I_Z = (y_1,\dots,y_k)$ and $I_H = (z_{k+1},\dots,z_n)$. Any $f \in \mathcal{O}_P$ can be expanded in a series $f = \sum_{\alpha \in \mathbb{N}^{n-k}} f_\alpha z^\alpha$ with $f_\alpha \in \mathcal{O}_H$. We set

$$f_o = f_{(0,\dots,0)} = f_{|H} \in \mathcal{O}_P$$
$$f_j = f_{\alpha_j} = \partial_{z_j} f_{|H} \in \mathcal{O}_P,$$

where $\alpha_j = (0,\dots,0,1,0,\dots,0)$ with 1 on the (j-k)-th place, $k+1 \le j \le n$, and $\mathcal{O}_H \subset \mathcal{O}_P$ is given canonically by the coordinates. Finally set $j_y(g) = (\partial_{y_1} g,\dots,\partial_{y_k} g)$ for $g \in \mathcal{O}_P$.

Proposition 1. (a) For X defined by f there are equivalent:

(1) Sing X is constant along Z w.r.t. (y,z).

(2) $(f_\alpha),\ j_y(f_\alpha) \subset (f_o) + (f_{k+1},\dots,f_n) + j_y(f_o)$ for all α.

(b) If Sing X is constant along Z w.r.t. (y,z) there are equivalent:

(1) X is trivial along Z.

(2) $f_\alpha \in (f_o) + I_Z \cdot j_y(f_o)$ for all α.

Proof. (a) By Theorem 1, (5), condition (1) means

$$Tf = (f_o) + (f_{k+1},\dots,f_n) + j_y(f_o). \qquad (*)$$

Comparison of coefficients implies (2). Conversely, (2) implies (*) by the Lemma applied to $I = (f_o) + (f_{k+1},\dots,f_n) + j_y(f_o)$ and $J = Tf$.

(b) Applying the Lemma to $I = (f) + I_Z \cdot j(f)$ and $J = (f) + I_Z \cdot (f_{k+1},\dots,f_n) + I_Z \cdot j_y(f_o)$ we get from (a) the equality of the two ideals. By Theorem 1, (4'), the triviality of X along Z is therefore equivalent to

$$\partial_{z_i} f \in (f) + I_Z \cdot (f_{k+1},\dots,f_n) + I_Z \cdot j_y(f_o) \qquad (\Delta)$$

for all i. Restricting to H, (Δ) implies

$$f_i \in (f_o) + I_Z \cdot (f_{k+1},\dots,f_n) + I_Z \cdot j_y(f_o).$$

Nakayama's Lemma then yields

$$f_i \in (f_o) + I_Z \cdot j_y(f_o)$$

and hence $\quad (f) + I_Z \cdot (f_{k+1},\dots,f_n) + I_Z \cdot j_y(f_o) \subset (f) + (f_o) + I_Z \cdot j_y(f_o).$

(Δ) implies $\quad \partial_{z_i} f \in (f) + (f_o) + I_Z \cdot j_y(f_o).$

Comparing coefficients one obtains inductively for all α that

$$f_\alpha \in (f_o) + I_Z \cdot j_y(f_o).$$

Conversely, if $f_\alpha \in (f_o) + I_Z \cdot j_y(f_o)$ apply the Lemma to $I = (f_o) + I_Z \cdot j_y(f_o)$ and $J = (f) + I_Z \cdot j_y(f_o)$ to obtain the equality of these two ideals. Condition (Δ) is then immediate. This proves the Proposition.

As a first application of the criterion we have a Bertini type statement concerning singular strata:

Proposition 2. Let $X \subset P$ be a hypersurface and $H \subset P$ smooth and transversal to Σ_X. Then $X \cap H$ is a hypersurface in H and

$$\Sigma_{X \cap H} \subset \Sigma_X \cap H.$$

Equality holds if H and Σ_X are strictly transversal or if $\mathrm{Sing}(X \cap H) = (\mathrm{Sing}\, X) \cap H$.

Proof. Fix a smooth $Z \subset \Sigma_X$ strictly transversal to H. By Theorem 1, (5'), choose coordinates (y,z) on P with $I_Z = (y_1,\dots,y_k)$, $I_H = (z_{k+1},\dots,z_n)$ and such that Sing X is constant along Z w.r.t. (y,z). Using the notation introduced above we have

$$Tf = (f_o) + (f_{k+1},\dots,f_n) + j_y(f_o) \qquad (1)$$

and $\quad (f_\alpha),\ j_y(f_\alpha) \subset Tf \quad$ for all α.

First assume $X \cap H = H$, i.e. $f_o = 0$. Then $Tf = (f_{k+1},\dots,f_n)$ and thus $\partial_{y_j}(Tf) \subset Tf$ for all j. Hence Sing X is trivial along H, i.e. $H \subset \Sigma_X$. By transversality we would have $\Sigma_X = P$, but this is absurd. Therefore $X \cap H$ is a hypersurface in H. Set $V = \Sigma_{X \cap H}$. By changing coordinates on H we may assume $I_V = (y_{m+1},\dots,y_k,z_{k+1},\dots,z_n)$ for some m and – by Theorem 1 – that $\mathrm{Sing}(X \cap H)$ is constant along V w.r.t. $y_1,\dots,y_k$.

Hence $\quad \partial_{y_j}(Tf_o) \subset Tf_o \quad$ for $m+1 \le j \le k$. $\qquad (2)$

(1) and (2) imply $\quad \partial_{y_j}(Tf) \subset Tf \quad$ for $m+1 \le j \le k$

and $\quad \partial_{z_j}(Tf) \subset Tf \quad$ for $k+1 \le j \le n$.

Therefore Sing X is trivial along V, i.e. $V \subset \Sigma_X$.

The equality is obvious if H and Σ_X are strictly transversal. Using Theorem 1, (5), one checks easily – since H is transversal to Σ_X – that triviality of Sing X along Σ_X implies triviality of $(\mathrm{Sing}\, X) \cap H$ along $\Sigma_X \cap H$. So equality follows again if $\mathrm{Sing}(X \cap H) = (\mathrm{Sing}\, X) \cap H$.

4. Infinitesimal Triviality and Harmony

As a motivation consider the following

Example. Let $g \in \mathbb{C}\{x_1,\dots,x_r\}$, $h \in \mathbb{C}\{y_1,\dots,y_s\}$ with $g \notin j(g)$, $h \notin j(h)$, and $u = 1 + v(z) \in \mathbb{C}\{z\}$, $0 \neq v(z) \in (z)$. Define X in $P = (\mathbb{C}^{r+s+1},0)$ by $f = g + u\cdot h$. Similarly as in [G-H, §4] the reader may check - using Proposition 1 - that in case $v(z)$ has non-vanishing linear term, X is dissonant along $Z = 0\times 0\times(\mathbb{C},0)$.

On the other hand we claim that dissonance along Z does not appear if $v(z) \in (z^2)$: One checks that $Tf = (g+h) + (v'\cdot h) + j(g) + j(h)$. If Sing X were trivial along Z, Theorem 1, (4'), would imply that

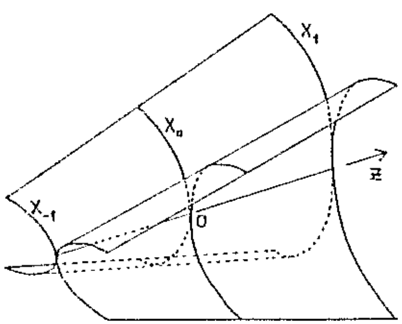
$$\partial_z(v'\cdot h) = v''\cdot h \in Tf + I_Z\cdot j(v'\cdot h) =$$
$$= (g+h) + j(g) + j(h) + m_P\cdot(v''\cdot h).$$

From Nakayama's Lemma and comparison of coefficients follows

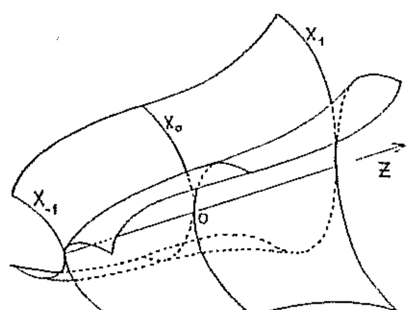
$$h \in (g+h) + j(g) + j(h).$$

Since $g \notin j(g)$ and $h \notin j(h)$, this is impossible. Hence X is not dissonant along Z.

Here is a geometric interpretation of the example: For $t \in \mathbb{C}$ let $H_t = \mathbb{C}^r\times\mathbb{C}^s\times\{t\}$ and $X_t = X\cap H_t$. Then X_t is the fibre product over $g : (\mathbb{C}^r,0)\to\mathbb{C}$ and $(1+v(t))\cdot h : (\mathbb{C}^s,0)\to\mathbb{C}$. Since $X = \bigcup_t X_t$, the factor $(1+v(t))$ can be interpreted as the ratio with which X stretches along Z. The cases $v(z) = z$ and $v(z) = z^3$ are presented below.



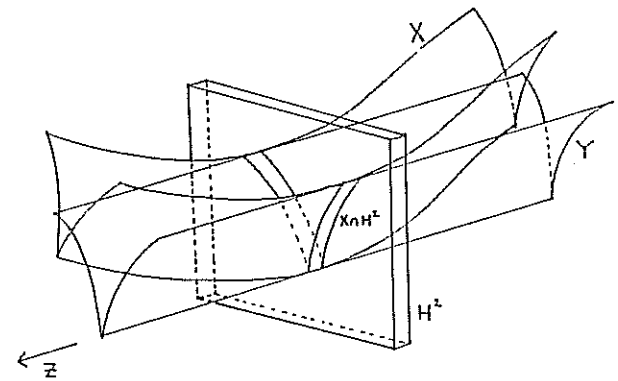
X dissonant along Z



X harmonic along Z

In the example, dissonance along Z only depends on the linear term of f w.r.t. the z-coordinate. This suggests the definition of infinitesimal triviality:

Definition. Let X be an arbitrary space germ embedded in smooth P. Let Z and H be strictly transversal smooth subspaces of P. X is called infinitesimally trivial along Z w.r.t. H if there is a germ $Y \subset P$ trivial along Z such that $X\cap H^2 = Y\cap H^2$. Here H^2 denotes the first infinitesimal neighborhood of H in P defined by the ideal $(I_H)^2$.



Remarks. (1) The space Y is not uniquely determined: Let $P = (\mathbb{C}^3,0)$ with coordinates x,y,z and set $I_X = (x^2+y^2)$, $I_Y = (x^2(1+z^2)^2+y^2)$, $I_Z = (x,y)$, $I_H = (z)$. Then X and Y are trivial along Z, $X\cap H^2 = Y\cap H^2$ but $X \neq Y$. However, Y is unique up to isomorphism, for $X\cap H^2 = Y\cap H^2$ implies $X\cap H = Y\cap H$ and by Theorem 1, Y is isomorphic to $(Y\cap H)\times Z$.

(2) If $H \subset X$, then $Y = P$ and then even $H^2 \subset X$. If $H \not\subset X$ and $X \subset P$ is a hypersurface, then Y is a hypersurface.

(3) The notion of infinitesimal triviality is not independent of H: Let x,y,z be coordinates on $P = (\mathbb{C}^3,0)$, define Z, H and H' by $I_Z = (x,y)$, $I_H = (z)$ and $I_{H'} = (y+z)$. Set $f(x,y,z) = g(x) + z^2\cdot h(y)$ with $g \in (x^2)$ and $h \neq 0$. Then one can show using Proposition 3 below that X is infinitesimally trivial along Z w.r.t. H but not w.r.t. H'.

Proposition 3. For $X \subset P$ and H, $Z \subset P$ strictly transversal and smooth the following conditions are equivalent:

(1) X is infinitesimally trivial along Z w.r.t. H.

(2) There exist vectorfields $D_{k+1},\dots,D_n$ on P tangent to Z such that $D_{k+1}(0),\dots,D_n(0)$ span T_oZ and $D_j(I_X) \subset I_X + I_H$ for all j.

(2') For any (some) coordinates (y,z) on P with $I_Z = (y_1,\dots,y_k)$ and $I_H = (z_{k+1},\dots,z_n)$ one has:

$$\partial_{z_j} f \in I_X \cdot \mathcal{O}_P^m + I_Z \cdot j(f) + I_H \cdot \mathcal{O}_P^m \quad \text{for all } j$$

(notation as in Th. 1, (4')). If X is a hypersurface, the condition becomes:

$$\partial_{z_j} f \in (f) + I_Z \cdot j(f) + I_H.$$

Proof. (1) ⇒ (2): Choose $Y \subset P$ trivial along Z such that $X \cap H^2 = Y \cap H^2$. By Theorem 1, (5'), there exist coordinates (y,z) on P with $I_Z = (y_1,\dots,y_k)$ and such that $\partial_{z_j}(I_Y) \subset I_Y$ for all j. The vectorfields $D_j = \partial_{z_j}$ satisfy (2) since $I_X + I_H{}^2 = I_Y + I_H{}^2$.

(2) ⇒ (1): We may assume that $n-k = \dim Z$. Let D denote the vector of components D_j. The hypotheses imply that there exist coordinates (y,z) on P with $I_Z = (y_1,\dots,y_k)$ and $I_H = (z_{k+1},\dots,z_n)$ and such that

$$D = A \cdot \partial_y + B \cdot \partial_z$$

for some $(n-k) \times k$ matrix A with entries in I_Z and an invertible $(n-k) \times (n-k)$ matrix B with entries in $\mathcal{O}_P$. Without loss of generality we may assume $B = \mathrm{Id}$, say $D = A \cdot \partial_y + \partial_z$. One checks that after replacing (y,z) by $(y-zA,z)$ the entries of the new A also belong to I_H. Let $\pi : P \to H$ denote the projection on the first k coordinates. The germ $Y = \pi^{-1}(X \cap H)$ is clearly trivial along Z. On the other hand condition (2) implies that $\partial_{z_j}(I_X) \subset I_X + I_H$. As I_X and I_Y are congruent modulo $(\Sigma z_i \partial_{z_i}(I_X) + I_H{}^2)$ by definition, the desired equality $X \cap H^2 = Y \cap H^2$ follows. Thus X is infinitesimally trivial along Z w.r.t. H.

(2) ⇔ (2'): This can be verified by a simple computation.

From now on X is a non-smooth hypersurface. Among the various possibilities of defining infinitesimal harmony the following turns out the most effective:

Definition. X is called infinitesimally harmonic if for some embedding $X \subset P$ X is infinitesimally trivial along its singular stratum Σ_X w.r.t. some smooth $H \subset P$ strictly transversal to Σ_X. Otherwise X is called infinitesimally dissonant.

Remark. It will follow from Theorem 3 below that an infinitesimally harmonic X is infinitesimally trivial along Σ_X w.r.t. any smooth, strictly Σ_X-transversal $H \subset P$ and any $X \subset P$.

Proposition 4. Let X be in P and let H, $Z \subset P$ be smooth and strictly transversal with $Z \subset \Sigma_X$. If X is not harmonic along Z but infinitesimally trivial along Z w.r.t. H then $X \cap H$ is a hypersurface in H and dissonant along a curve C in H.

Proof. Choose coordinates (y,z) on P such that $I_Z = (y_1,\dots,y_k)$, $I_H = (z_{k+1},\dots,z_n)$ and Sing X is constant along Z w.r.t. (y,z). Set $f = \Sigma f_\alpha z^\alpha$, $f_\alpha \in \mathcal{O}_H$, $f_o = f_{|H}$ and $f_j = \partial_{z_j} f_{|H}$ as above. Proposition 3 implies

$$f_j \in (f_o) + I_Z \cdot (f_{k+1},\dots,f_n) + I_Z \cdot j_y(f_o) \quad \text{for all } j,$$

hence by Nakayama's Lemma

$$f_j \in (f_o) + I_Z \cdot j_y(f_o) \quad \text{for all } j.$$

Proposition 1, (a), yields

$$(f_\alpha),\ j_y(f_\alpha) \subset (f_o) + j_y(f_o) \quad \text{for all } \alpha. \qquad (*)$$

Since X is not trivial along Z there exists by Proposition 1, (b), an index β such that

$$f_\beta \notin (f_o) + I_Z \cdot j_y(f_o).$$

By (*) we can write: $f_\beta = h \cdot f_o + Df_o$

with $h \in \mathcal{O}_H$ and a derivation D of $\mathcal{O}_H$ not vanishing at the origin of H. By changing the coordinates on H we may assume that $D = \partial_{y_1}$. Then using (*) we get

$$\partial_{y_1}(j_y(f_o)) \subset j_y(f_\beta) + (f_o) + j_y(f_o) \subset (f_o) + j_y(f_o)$$

and

$$\partial_{y_1}(f_o) \notin (f_o) + (y_2,\dots,y_k) \cdot j_y(f_o).$$

By Theorem 1, (4'), this implies that Sing$(X \cap H)$ is trivial but $X \cap H$ is not trivial along the curve $C \subset H$ defined by $y_2 = \dots = y_k = 0$. This proves the assertion.

Example. In order to illustrate the situation described in the Proposition consider the following singularity: Let $P = (\mathbb{C}^{r+s+2},0) = (\mathbb{C}^{r+s+1},0) \times (\mathbb{C},0)$ with coordinates x,y,v,z and $H = (\mathbb{C}^{r+s+1},0) \times 0$, $Z = 0 \times (\mathbb{C},0) \subset P$, $C = 0 \times (\mathbb{C},0) \subset H$. Let $g \in \mathbb{C}\{x\}$ and $h \in \mathbb{C}\{y\}$ with $g \notin j(g)$ and $h \notin j(h)$ and define X in P by $f = g + (1+v)\cdot h + z^2\cdot(g-h)$. With the notation of Proposition 1 we have:

$$n = r+s+2,\ k = r+s+1,$$
$$f_o = g + (1+v)\cdot h,$$
$$Tf = (f_o) + j(f_o) = (g) + (h) + j(g) + j(h),$$
$$(f_o) + I_Z\cdot j(f_o) = (g+h) + I_Z\cdot[j(g) + j(h) + (h)].$$

We claim that X is not harmonic along Z, but infinitesimally trivial along Z w.r.t. H, and that $X \cap H$ is dissonant along the curve C in H. Indeed, since

$$g-h \notin (f_o) + I_Z\cdot j(f_o)$$

Proposition 1 implies that X is not harmonic along Z. It is clear from the definition that X is infinitesimally trivial along Z w.r.t. H. Finally, $X \cap H$ is defined in H by $g + (1+v)\cdot h$ and is – as we have seen earlier – dissonant along $C \subset H$.

Remark. As the next theorem shows the phenomenon that X is infinitesimally trivial but not trivial along Z is only possible if Z is strictly contained in Σ_X. In case $Z = \Sigma_X$ infinitesimal triviality along Z implies triviality along Z:

Theorem 3. A hypersurface singularity X is harmonic if and only if it is infinitesimally harmonic.

Proof. One implication is obvious. Assume conversely that X is infinitesimally harmonic. Then there exist an embedding $X \subset P$ and a smooth $H \subset P$ strictly transversal to Σ_X such that X is infinitesimally trivial along Σ_X w.r.t. H. By Proposition 2 the singular stratum $\Sigma_{X\cap H}$ of $X \cap H$ is 0. In particular, $X \cap H$ is harmonic. By Proposition 4, X is harmonic.

5. Quasi-homogeneous Singularities

Theorem 4. Let X be a quasi-homogeneous hypersurface singularity defined in smooth P by f = 0. Assume that the partial derivatives of f have non-zero degrees (i.e.: there are coordinates $u_1,\dots,u_n$ on P and $\lambda,\lambda_1,\dots,\lambda_n \in \mathbb{C}$ with $\lambda - \lambda_i \neq 0$ for all i such that $\lambda\cdot f = \Sigma\, \lambda_i u_i \partial_{u_i} f$). Then X is harmonic.

Proof. Consider the derivation $D = \Sigma\, \lambda_i u_i \partial_{u_i}$ of $\mathcal{O}_P$ and the induced one-parameter group Γ of automorphisms of P. Since Γ stabilizes X, it also stabilizes Sing X and $\Sigma = \Sigma_X$. We therefore have $D(I_\Sigma) \subset I_\Sigma$. Let (y,z) be coordinates on P with $I_\Sigma = (y_1,\dots,y_k)$ and such that Sing X is constant along Σ w.r.t. (y,z). Write in these coordinates $D = \Sigma\, a_i\partial_{y_i} + \Sigma\, b_i\partial_{z_i}$ with $a_i,\ b_i \in \mathcal{O}_P$. Since $D(I_\Sigma) \subset I_\Sigma$, the a_i belong to I_Σ and therefore $\partial_{z_j} a_i \in I_\Sigma$ for all i and j. Consider the matrices

$$\partial_y A = (\partial_{y_j} a_i) \qquad \partial_y B = (\partial_{y_j} b_i)$$
$$\partial_z A = (\partial_{z_j} a_i) \qquad \partial_z B = (\partial_{z_j} b_i).$$

The eigenvalues of $\begin{pmatrix}\partial_y A & \partial_y B\\ \partial_z A & \partial_z B\end{pmatrix}(0) = \begin{pmatrix}\partial_y A & \partial_y B\\ 0 & \partial_z B\end{pmatrix}(0)$

are just $\lambda_1,\dots,\lambda_n$, which are by assumption different from λ. Therefore the eigenvalues of $\partial_z B(0)$ are different from λ and the matrix $\lambda\cdot Id - \partial_z B$ is invertible. From $\lambda\cdot f = Df$ follows for all j

$$\lambda\cdot\partial_{z_j} f = \sum_i (\partial_{z_j} a_i\cdot\partial_{y_i} f + a_i\cdot\partial_{z_j}\partial_{y_i} f) + \sum_i (\partial_{z_j} b_i\cdot\partial_{z_i} f + b_i\cdot\partial_{z_j}\partial_{z_i} f)$$

hence $$(\lambda.Id - \partial_z B)\cdot(\partial_{z_{k+1}} f,\dots,\partial_{z_n} f)^T \equiv 0 \mod \mathfrak{m}_P\cdot Tf.$$

Therefore $\partial_{z_j} f \in (f) + \mathfrak{m}_P\cdot j(f)$ for all j. This implies $\dim A_X \geq n - k$ (in fact: if $k = n-1$ then $Df \in (f)$ with a derivation D of $\mathcal{O}_P$ not vanishing at 0 and $\dim A_X \geq 1$ by Theorem 1. The general case follows by induction on n-k). Thus $A_X = \Sigma_X$.

Example. We conclude the paper with an example of a dissonant quasi-homogeneous singularity: Let $g \in \mathbb{C}\{x_1,\dots,x_{r-2}\}$ with $g \notin j(g)$ and put $G = g + x_{r-1}x_r \in \mathbb{C}\{x_1,\dots,x_r\}$. If $x_1,\dots,x_r$ are given the weights 0,...,0,1,-1 then G is quasi-homogeneous of degree 0, but $G \notin j(G)$. Define X in $P = (\mathbb{C}^{2r+1},0)$ by $f = G(x) + (1+z)\cdot G(y) \in \mathbb{C}\{x,y,z\}$. Giving z weight 0, f is quasi-homogeneous, and X is dissonant as we have seen earlier.

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